On the game chromatic number of sparse random graphs

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Abstract

Given a graph G and an integer k, two players take turns coloring the vertices of G one by one using k colors so that neighboring vertices get different colors. The first player wins iff at the end of the game all the vertices of G are colored. The game chromatic number $\chi_g(G)$ is the minimum k for which the first player has a winning strategy. The paper [6] began the analysis of the asymptotic behavior of this parameter for a random graph $G_{n,p}$. This paper provides some further analysis for graphs with constant average degree i.e. $np = O(1)$ and for random regular graphs.

1   Introduction

Let $G = (V, E)$ be a graph and let k be a positive integer. Consider the following game in which two players A(lice) and B(ob) take turns in coloring the vertices of G with k colors. Each move consists of choosing an uncolored vertex of the graph and assigning to it a color from $\{1, \ldots, k\}$ so that the resulting coloring is *proper*, i.e., adjacent vertices get different colors. A wins if all the vertices of G are eventually colored. B wins if at some point in the game the current partial coloring cannot be extended to a complete coloring of G, i.e., there is an uncolored vertex such that each of the k colors appears at least once in its neighborhood. We assume that A goes first (our results will not be sensitive to this choice). The *game chromatic number* $\chi_g(G)$ is the least integer k for which A has a winning strategy.

This parameter is well defined, since it is easy to see that A always wins if the number of colors is larger than the maximum degree of G. Clearly, $\chi_g(G)$ is at least as large as the ordinary chromatic number $\chi(G)$, but it can be considerably more. The game was first considered by Brams about 25 years ago in the context of coloring planar graphs and was described in Martin Gardner's column [12] in Scientific American in 1981. The game remained unnoticed by the graph-theoretic

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community until Bodlaender [5] re-invented it. For a survey see Bartnicki, Grytczuk, Kierstead and Zhu [4].

In this paper, we study the game chromatic number of the random graph $G_{n,p}$ and the random d-regular graph $G_{n,d}$. Define $b = \frac{1}{1-p}$. The following estimates were proved in Bohman, Frieze and Sudakov [6].

Theorem 1.1.

(a) *There exists $K > 0$ such that for $\varepsilon > 0$ and $p \geq (\ln n)^{K\varepsilon^{-3}}/n$ we have that w.h.p.*[1]

$$\chi_g(G_{n,p}) \geq (1 - \varepsilon)\frac{n}{\log_b np}\ .$$

(b) *If $\alpha > 2$ is a constant, $K = \max\{\frac{2\alpha}{\alpha-1}, \frac{\alpha}{\alpha-2}\}$ and $p \geq (\ln n)^K/n$ then w.h.p.*

$$\chi_g(G_{n,p}) \leq \alpha\frac{n}{\log_b np}\ .$$

In this paper we complement these results by considering the case where $p = \frac{d}{n}$ where d is at least some sufficiently large constant. We will assume that $d \leq n^{1/4}$ since Theorem 1.1 covers larger d.

Theorem 1.2. *Let $p = \frac{d}{n}$ where d is larger than some absolute constant and $d \leq n^{1/4}$.*

(a) *If $\alpha < \frac{4}{7}$ is a constant then w.h.p.*

$$\chi_g(G_{n,p}) \geq \frac{\alpha d}{\ln d}.$$

(b) *If $\alpha > 6$ is a constant then w.h.p.*

$$\chi_g(G_{n,p}) \leq \frac{\alpha d}{\ln d}.$$

Note that when $p = o(1)$ we have $\frac{n}{\log_b np} \sim \frac{d}{\ln d}$. Note also that the bounds in Theorem 1.1 are stronger than those in Theorem 1.2, whenever both results are applicable.

It is natural to compare our bounds with the asymptotic behavior of the ordinary chromatic number of random graph. It is known by the results of Bollobás [7] and Łuczak [16]) that when $p = o(1)$, $\chi(G_{n,p}) = (1 + o(1))\frac{d}{2\ln d}$ w.h.p.. (Of course a stronger result is now known, see Achlioptas and Naor [2]). Thus Theorem 1.2 shows that the game chromatic number of $G_{n,p}$ is at most (roughly) twelve times and at least (roughly) 8/7 times its chromatic number.

Having proved Theorem 1.2, we extend the results to the random d-regular graph $G_{n,d}$.

Theorem 1.3. *Let $\varepsilon > 0$ be an arbitrary constant.*

(a) *If α is a constant satisfying the conditions of Theorem 1.1 or Theorem 1.2 where appropriate and d is sufficiently large and $d \leq n^{1/3-\varepsilon}$ then w.h.p.*

$$\chi_g(G_{n,d}) \geq \frac{\alpha d}{\ln d}\ .$$

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[1]A sequence of events $\mathcal{E}_n$ occurs *with high probability* (w.h.p.) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$

(b) *If α is a constant satisfying the conditions of Theorem 1.1 or Theorem 1.2 where appropriate and d is sufficiently large and $d \leq n^{1/3-\varepsilon}$ then w.h.p.*

$$\chi_g(G_{n,d}) \leq \frac{\alpha d}{\ln d}.$$

It is known by the result of Frieze and Łuczak [11]) that w.h.p. $\chi(G_{n,d}) = (1 + o(1))\frac{d}{2\ln d}$. (Of course stronger results are now known, see Achlioptas and Moore [1] and Kemkes, Péres-Giménez and Wormald [14]).

Theorem 1.3 says nothing about $\chi_g(G_{n,d})$ when d is small. We have been able to prove

Theorem 1.4. *If $d = 3$ then w.h.p. $\chi(G_{n,d}) = 4$.*

It is easy to see via Brook's theorem that w.h.p. the chromatic number of a random cubic graph is three and so Theorem 1.4 seperates χ and χ_g in this context.

We often refer to the following Chernoff-type bounds for the tails of binomial distributions (see, e.g., [3] or [13]). Let $X = \sum_{i=1}^{n} X_i$ be a sum of independent indicator random variables such that $\mathbb{P}(X_i = 1) = p_i$ and let $p = (p_1 + \cdots + p_n)/n$. Then

$$\mathbb{P}(X \leq (1 - \varepsilon)np) \leq e^{-\varepsilon^2 np/2}, \tag{1.1}$$

$$\mathbb{P}(X \geq (1 + \varepsilon)np) \leq e^{-\varepsilon^2 np/3}, \qquad \varepsilon \leq 1, \tag{1.2}$$

$$\mathbb{P}(X \geq \mu np) \leq (e/\mu)^{\mu np}. \tag{1.3}$$

1.1 Outline of the paper

Section 2 is devoted to the proof of Theorem 1.2. In Section 2.1, we prove a lower bound on $\chi_g(G_{n,p})$ by giving a strategy for player B. Basically, B's strategy is to follow A coloring a vertex with color i by coloring a random vertex v with color i. Of course we mean here that v is randomly chosen from vertices outside of the neighborhood of the set of vertices of color i. Why does this work? Well, it is known that choosing an independent set via a greedy algorithm will w.h.p. find an independent set that is about one half the size of the largest independent set. What we show is that choosing randomly half the time also has a deleterious effect on the size of the independent set (color class) selected. This leads to the game chromatic number being significantly larger than the chromatic number.

In Section 2.2, we prove an upper bound on $\chi_g(G_{n,p})$ by giving a strategy for player A. Here A follows the same strategy used in the proof of Theorem 1.1(b), up until close to the end. We then let A follow a more sophisticated strategy. A's initial strategy is to choose a vertex with as few "available" colors and color it with any available color i.e. one that does not conflict with its colored neighbors. At a certain point there are few uncolored vertices and they all have a substantial number of available colors. We show that the edges of the graph induced by these vertices can be partitioned into a forest F plus a low degree subgraph. Using the tree coloring strategy described in [10] we see that the low degree subgraph does not prevent G from being colored.

Having proved Theorem 1.2 we transfer the results to random regular graphs by showing that the underlying structural lemmas remain true or trivially modified. This is done in Section 3.

In section 4 we provide a strategy for B showing that w.h.p. $\chi_g(G_{n,3}) = 4$. This proves Theorem 1.4. B's strategy is based on his ability to force A into playing on a small set of vertices. B will then make a sequence of such forcing moves along a cycle to create a double threat and win the game.

In Appendix A we complete the picture by showing how to convert Theorem 1.1 into a random regular graph setting using the "Sandwiching Theorem" of Kim and Vu [15].

2 Theorem 1.2: $G_{n,p}, p = d/n$

2.1 The lower bound

Let $D = \frac{\ln d}{d}$ and suppose that there are $k = \alpha/D$ colors. At any stage, let C_i be the set of vertices that have been colored i and let $C = \bigcup_{i=1}^{k} C_i$. Let $U = [n] \setminus C$ be the set of uncolored vertices and let $U_i = U \setminus N(C_i)$. Note that $[n] = \{1, 2, \ldots, n\}$ is the vertex set of $G_{n,p}$.

B's strategy will be to choose the same color that A just chose and then assign color i to a random vertex in U_i. The idea here being that making random choices when constructing an independent set (color class) tends to only get one of half the maximum size. A could be making better choices and so we do not manage to prove that we need twice as many colors as the minimum.

Suppose that we run this process for θn rounds and that $|C_i| = 2\beta Dn$ where we will later take $\theta = 1/2 - \varepsilon$ and $\beta = 1/2$ for some small fixed $\varepsilon > 0$. Let S_i be the set of βDn vertices in C_i that were colored by B. We consider the probability that there exists a set T of size γDn such that $C_i \cup T$ is independent. We use the notation $A \leq_b B$ for $A = O(B)$ when the bracketing is "ugly".

$$\mathbb{P}(\exists C_i, T) \leq_b \binom{n}{\beta Dn} \sum_{|S|=\beta Dn} \mathbb{P}(S_i = S) \binom{n}{\gamma Dn}(1-p)^{(2\beta+\gamma)^2 D^2 n^2/2} \tag{2.1}$$

$$\leq \binom{n}{\beta Dn} \sum_{|S|=\beta Dn} (\beta Dn)! \prod_{j=1}^{\beta Dn} \frac{2}{(1-p)^{2j}(1-2\theta)n} \binom{n}{\gamma Dn}(1-p)^{(2\beta+\gamma)^2 D^2 n^2/2} \tag{2.2}$$

$$\leq \binom{n}{\beta Dn}^2 \frac{(\beta Dn)!}{((1-2\theta)n)^{\beta Dn}} \frac{2^{\beta Dn}}{(1-p)^{\beta^2 D^2 n^2}} \binom{n}{\gamma Dn}(1-p)^{(2\beta+\gamma)^2 D^2 n^2/2}$$

$$\leq \left(\left(\frac{e}{\beta D}\right)^{\beta} \cdot \left(\frac{2}{1-2\theta}\right)^{\beta} \cdot \left(\frac{e}{\gamma D}\right)^{\gamma} \cdot \exp\left\{(\beta^2 - (2\beta+\gamma)^2)dD/2\right\} \right)^{Dn}$$

$$= \exp\left\{(\beta + \gamma + \beta^2 - (2\beta+\gamma)^2/2 + o_d(1)))Dn \ln d\right\} \tag{2.3}$$

$$= o(1)$$

if $(2\beta+\gamma)^2 > 2(\beta + \beta^2 + \gamma)$. This is satisfied when $\beta = 1/2$ and $\gamma = 3/4$. We will justify (2.1) and (2.2) momentarily.

If the event $\{\exists C_i, T\}$ does not occur then the number ℓ of colors i for which $|S_i| \geq \beta Dn$ by this time satisfies

$$\ell(2\beta+\gamma)D + 2(k-\ell)\beta D \geq 2\theta.$$

We choose $\theta = 1/2 - \varepsilon$ where $\varepsilon > 0$ is an arbitrarily small positive constant. Since $k \geq \ell$, this

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implies that
$$kD \geq \frac{2\theta}{2\beta + \gamma} = \frac{4(1 - 2\varepsilon)}{7}.$$

This completes the proof of Part (a) of Theorem 1.2.

Justifying (2.1)**:** Here we are taking the union bound over all $\binom{n}{\beta Dn}\binom{n}{\gamma Dn}$ possible choices of $C_i \setminus S_i$ and T.

Justifying (2.2)**:** For this we first consider a sequence of random variables $X_0 = N, X_j = X_{j-1} \cdot \text{Bin}(X_{j-1}, q)$ for $1 \leq j \leq t$. Then we estimate $\mathbb{E}(Y_t)$ where

$$Y_t = \begin{cases} 0 & X_t = 0 \\ \frac{1}{X_0 X_1 \cdots X_t} & X_t > 0 \end{cases}$$

With $N = (1 - 2\theta)n$ we interpret $\frac{1}{X_0 X_1 \cdots X_t}$ as (a bound on) the probability that $S = S_i$, given the number of vertices X_j available for coloring with i after $2j$ vertices have already been colored. We take the expectation over $G_{n,p}$ and hence use $q = (1 - p)^2$.

Now if $B = B(\nu, q)$ and we take $\prod_{i=1}^{k} \frac{1}{B+i-1} = 0$ when $B = 0$ then

$$\mathbb{E}\left(\prod_{i=1}^{k} \frac{1}{B + i - 1} \right) = \sum_{\ell=1}^{\nu} \prod_{i=1}^{k} \frac{1}{\ell + i - 1} \binom{\nu}{\ell} q^\ell (1 - q)^{\nu - \ell}$$

$$\leq \frac{2}{q^k} \prod_{i=1}^{k} \frac{1}{\nu + i} \sum_{\ell=0}^{\nu} \binom{\nu + k}{\ell + k} q^{\ell+k}(1 - q)^{\nu - \ell}$$

$$\leq \frac{2}{q^k} \prod_{i=1}^{k} \frac{1}{\nu + i}.$$

It follows that

$$\mathbb{E}\left(\frac{1}{X_0 X_1 \cdots X_t} \right)$$
$$\leq \mathbb{E}\left(\frac{2}{X_0 X_1 \cdots X_{t-1}(X_{t-1} + 1)q} \right)$$
$$\leq \mathbb{E}\left(\frac{2^2}{X_0 X_1 \cdots X_{t-2}(X_{t-2} + 1)(X_{t-2} + 2)q^{1+2}} \right)$$
$$\vdots$$
$$\leq \frac{2^t}{N(N + 1) \cdots (N + t)q^{1+2+\cdots+t}}.$$

2.2   The upper bound

We begin by proving some simple structural properties of $G_{n,p}$.

Lemma 2.1. *If* $\theta > 1$ *and*

$$\left(\frac{\sigma e d}{2\theta} \right)^\theta \leq \frac{\sigma}{2e} \tag{2.4}$$

then w.h.p. there does not exist $S \subseteq [n], |S| \leq \sigma n$ *such that* $e(S) \geq \theta|S|$.

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Proof

$$\mathbb{P}(\exists S : |S| \le \sigma n \text{ and } e(S) \ge \theta|S|) \le \sum_{s=2\theta}^{\sigma n} \binom{n}{s} \binom{\binom{s}{2}}{\theta s} \left(\frac{d}{n}\right)^{\theta s} \tag{2.5}$$

$$\le \sum_{s=2\theta}^{\sigma n} \left(\frac{ne}{s} \left(\frac{eds}{2\theta n}\right)^{\theta}\right)^{s}$$

$$= \sum_{s=2\theta}^{\sigma n} \left(e \left(\frac{s}{n}\right)^{\theta-1} \left(\frac{ed}{2\theta}\right)^{\theta}\right)^{s} \tag{2.6}$$

$$= O\left(\frac{d^{\theta}}{n^{\theta-1}}\right) = o(1)$$

provided $d = o(n^{1-1/\theta})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will apply this lemma with $\theta \ge 2 - \varepsilon$ for $\varepsilon \ll 1$ and this fits with our bound on d.

Lemma 2.2. *Let σ, θ be as in Lemma 2.1. If $(\delta - 2\theta)\tau > 1$ and*

$$\left(\frac{\sigma e d}{(\delta - 2\theta)\tau}\right)^{(\delta-2\theta)\tau} \le \frac{\sigma}{4e}$$

then w.h.p. there do not exist $S \supseteq T$ such that $|S| \le \sigma n, |T| \ge \tau s$ and $d_S(v) \ge \delta$ for $v \in T$.

Proof   In the light of Lemma 2.1, the assumptions imply that w.h.p. $|e(T : S \backslash T)| \ge (\delta - 2\theta)\tau s$,

$$\mathbb{P}(\exists S \supseteq T, |S| \le \sigma n, |T| \ge \tau s : |e(T : S \setminus T)| \ge (\delta - 2\theta)\tau s)$$

$$\le \sum_{s=2\theta}^{\sigma n} \sum_{t=\tau s}^{s} \binom{n}{s} \binom{s}{t} \left(\frac{edt}{(\delta - 2\theta)\tau n}\right)^{(\delta-2\theta)\tau s} \tag{2.7}$$

$$\le \sum_{s=2\theta}^{\sigma n} \sum_{t=\tau s}^{s} \left(\frac{ne}{s}\right)^{s} \cdot 2^{s} \cdot \left(\frac{eds}{(\delta - 2\theta)\tau n}\right)^{(\delta-2\theta)\tau s}$$

$$= \sum_{s=2\theta}^{\sigma n} \sum_{t=\tau s}^{s} \left(\frac{2ne}{s} \cdot \left(\frac{eds}{(\delta - 2\theta)\tau n}\right)^{(\delta-2\theta)\tau}\right)^{s}$$

$$= \sum_{s=2\theta}^{\sigma n} \sum_{t=\tau s}^{s} \left(2e \left(\frac{s}{n}\right)^{(\delta-2\theta)\tau-1} \cdot \left(\frac{ed}{(\delta - 2\theta)\tau}\right)^{(\delta-2\theta)\tau}\right)^{s} \tag{2.8}$$

$$= O\left(\frac{d^{(\delta-2\theta)\tau}}{n^{(\delta-2\theta)\tau-1}}\right) = o(1).$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$$

We will apply this lemma with $(\delta - 2\theta)\tau \ge 2$ and this fits with out bound on d.

Fix $\alpha > 1$ and let

$$k = \frac{\alpha d}{\ln d} \text{ and } \beta = \frac{\alpha d^{1-1/\alpha}}{\ln d} \text{ and } \gamma = \frac{16 \ln^2 d}{\alpha d^{1-1/\alpha}}.$$

We will now argue that w.h.p. A can win the game if k colors are available.

A's initial strategy will be the same as that described in [6]. Let $\mathcal{C} = (C_1, C_2, \ldots, C_k)$ be a collection of pair-wise disjoint sub-sets of $[n]$, i.e. a (partial) coloring. Let $\bigcup \mathcal{C}$ denote $\bigcup_{i=1}^{k} C_i$. For a vertex v let

$$A(v, \mathcal{C}) = \{i \in [k] : v \text{ is not adjacent to any vertex of } C_i\},$$

and set

$$a(v, \mathcal{C}) = |A(v, \mathcal{C})|.$$

Note that $A(v, \mathcal{C})$ is the set of colors that are available at vertex v when the partial coloring is given by the sets in $\mathcal{C}$ and $v \notin \bigcup \mathcal{C}$. A's initial strategy can now be easily defined. Given the current color classes $\mathcal{C}$, A chooses an uncolored vertex v with the smallest value of $a(v, \mathcal{C})$ and colors it by any available color.

As the game evolves, we let u denote the number of uncolored vertices in the graph. So, we think of u as running "backward" from n to 0.

We show next that w.h.p. every coloring of the full vertex set has the property that there are at most γn vertices with less than $\beta/2$ available colors. Let

$$B(\mathcal{C}) = \{v : a(v, \mathcal{C}) < \beta/2\}.$$

Lemma 2.3. *W.h.p., for all collections $\mathcal{C}$,*

$$|B(\mathcal{C})| \leq \gamma n.$$

Proof Fix $\mathcal{C}$. Then for every $v \notin \bigcup \mathcal{C}$, the number of colors available at v is the sum of independent indicator variables X_i, where $X_i = 1$ if v has no neighbors in C_i. Then $\mathbb{P}(X_i = 1) = (1-p)^{|C_i|}$ and since $(1-p)^t$ is a convex function of t we have

$$
\begin{aligned}
\mathbb{E}(a(v, \mathcal{C})) &= \sum_{i=1}^{k}(1-p)^{|C_i|} \\
&\geq k(1-p)^{(|C_1|+\cdots+|C_k|)/k} \\
&\geq k(1-p)^{n/k} = \beta.
\end{aligned}
$$

It follows from the Chernoff bound (1.1) that

$$\mathbb{P}(a(v, \mathcal{C}) \leq \beta/2) \leq e^{-\beta/8}.$$

Thus,

$$
\begin{aligned}
&\mathbb{P}(\exists \mathcal{C} : |B(\mathcal{C})| \geq \gamma n) \\
&\leq k^n \binom{n}{\gamma n} e^{-\beta\gamma n/8} \\
&\leq d^n \left(\frac{e}{\gamma} \exp\left\{-\frac{\alpha d^{1-1/\alpha}}{8 \ln d}\right\}\right)^{\gamma n} \\
&= \exp\left\{n\left(\ln d + \gamma\left(1 + \ln\alpha/16 + (1-1/\alpha)\ln d - 2\ln\ln d - \frac{\alpha d^{1-1/\alpha}}{8\ln d}\right)\right)\right\} \\
&= o(1)
\end{aligned}
$$

(2.9)

for large d. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Let u_0 to be the last time for which A colors a vertex with at least $d_0 = \beta/2$ available colors, i.e.,

$$u_0 = \min \left\{ u : \; a(v, \mathcal{C}_u) \geq d_0 = \beta/2, \text{ for all } v \notin \bigcup \mathcal{C}_u \right\},$$

where $\mathcal{C}_u$ denotes the collection of color classes when u vertices remain uncolored.

If u_0 does not exist then A will win.

It follows from Lemma 2.3 that w.h.p. $u_0 \leq \gamma n$. This implies that at some point where the number of uncolored vertices is less than γn, every vertex still has at least $d_0 = \beta/2$ available colors. At this point the set of uncolored vertices U satisfies $|U| \leq \gamma n$. A will now follow a more sophisticated strategy. We will show next that we can find a sequence $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ with the following properties: The edges of $U_i : (U_{i-1} \setminus U_i)$ between U_i and $U_{i-1} \setminus U_i$ are colored red or they are uncolored.

(P1) Each vertex of $U_i \setminus U_{i+1}$ has at most one uncolored neighbor in U_{i+1}, for $0 \leq i < \ell$.

(P2) All $U_i : (U_{i-1} \setminus U_i)$ edges are uncolored for $i \geq 2$.

(P3) Each vertex of U_1 has at most $\beta/10$ red neighbors in $U_0 \setminus U_1$.

(P4) $d_{U_i}(v) \leq \beta/4$ for $v \in U_i \setminus U_{i+1}$.

(P5) U_ℓ contains at most one cycle.

From this, we can deduce that the edges of U_0 can be divided up into the red edges R, uncolored edges F, the $U_i : U_{i+1}$, $i < \ell$ edges plus the edges inside U_ℓ and a remainder E_0. Assume first that U_ℓ does not contain a cycle. F is a forest and the strategy in [10] can be applied. When attempting to color a vertex v of F, there are never more than three F-neighbors of v that have been colored. Since there are at most $\beta/4 + \beta/10$ non-F neighbors, A will succeed since he/she has an initial list of size $\beta/2$.

If U_ℓ contains a cycle C then A can begin by coloring a vertex of C. This puts A one move behind in the tree coloring strategy, in which case we can bound the number of F-neighbors by four.

It only remains to prove that the construction P1–P5 exists w.h.p. Remember that d is sufficiently large here.

We can assume without loss of generality that $|U_0| = \gamma n$. This will not decrease the sizes of the sets $a(v, U_0)$.

2.2.1 The verification of P1–P4: Constructing U_1

Applying Lemma 2.2 with

$$\sigma = \gamma \text{ and } \theta = d^{1/\alpha} \ln^3 d \text{ and } \delta = 2\theta + \beta/4 \text{ and } \tau = \theta/\beta$$

we see that w.h.p.

$$U'_{1,a} = \left\{ v \in U_0 : d_{U_0}(v) \geq 2d^{1/\alpha} \ln^3 d + \beta/4 \right\} \text{ satisfies } |U'_{1,a}| \leq \tau \gamma n = \frac{16 d^{3/\alpha} \ln^6 d}{\alpha^2 d^2} n.$$

We then let $U_{1,a} \supseteq U'_{1,a}$ be the subset of U_0 consisting of the vertices with the $\tau\gamma n$ largest values of d_{U_0}.

We then construct $U_{1,b} \supseteq U_{1,a}$ by repeatedly adding vertices $x_1, x_2, \ldots, x_r$ of $U \setminus U_{1,a}$ such that x_j is the lowest numbered vertex not in $X_j = U_{1,a} \cup \{x_1, x_2, \ldots, x_{j-1}\}$ having at least three neighbors in X_i. This ends with $r \le 5|U_{1,a}|$ in order that we do not violate the conclusion of Lemma 2.1 with

$$\sigma = 6\tau\gamma = \frac{96d^{3/\alpha}\ln^6 d}{\alpha^2 d^2} \text{ and } \theta = 5/2 \text{ which is applicable since } \left(\frac{96ed^{3/\alpha}\ln^6 d}{5\alpha^2 d}\right)^{5/2} < \frac{80d^{3/\alpha}\ln^6 d}{2e\alpha^2 d^2}.$$

Every vertex in $U_{1,a} \setminus U_{1,b}$ has at most two neighbors in $U_{1,b}$ and we claim that the distribution of these pairs of neighbors is independent and uniform. To see this suppose that $u \in U_{1,a} \setminus U_{1,b}$ has neighbors y_1, y_2 in $U_{1,b}$ and we change one of the neighbors to z and re-run the construction of $U_{1,b}$. We claim that $U_{1,b}$ will be unchanged. This is because the change from (u, y_1) to (u, z) will not change the count of the number of neighbors of any x_j in X_j. This verifies the claim because when building $U_{1,b}$ we will never ask for the neigbors of u in an X_j, only the count.

Next let A be the set of vertices in $U_{1,a} \setminus U_{1,b}$ that have two neighbors in $U_{1,b}$ and let B be the set of vertices in $U_{1,b}$ that have more than $\beta/20$ neighbors in A. For a fixed $x \in U_{1,b}$ we have

$$\mathbb{P}(x \in B) \le \binom{2\gamma n}{\beta/20} \frac{1}{(\tau\gamma n)^{\beta/20}} \le \left(\frac{40e}{\tau\beta}\right)^{\beta/20} \le d^{-\beta/20\alpha}.$$

The events $x \in B$ and $x' \in B$ are negatively correlated and so the size of B is dominated by $\text{Bin}(\tau\gamma n, d^{-\beta/20\alpha})$. It follows that w.h.p.

$$|B| \le \tau\gamma d^{-\beta/40\alpha} n.$$

The edges $(U_1 \setminus B) : (U_1 \setminus U_0)$ are colored red.

Note that $B = \emptyset$ w.h.p. if $d \ge \ln^2 n$.

Now let A_1 be the set of neighbors of B in U_0. Next let

$$D_G = \sum_{\substack{v \in [n] \\ d_G(v) \ge 3d}} d_G(v).$$

Then w.h.p.

$$|A_1| \le D_G + 3d|B| \le D_G + 3d\tau\gamma d^{-\beta/40\alpha} n.$$

Now $D_G = 0$ w.h.p. if $d \ge \ln^2 n$ and otherwise we have

$$\mathbb{E}(D_G) = n \sum_{k=3d}^{n-1} u_k \text{ where } u_k = k\binom{n-1}{k}\left(\frac{d}{n}\right)^k \left(1 - \frac{d}{n}\right)^{n-1-k}.$$

Now $u_{k+1}/u_k \le 1/2$ and so

$$E(D_G) \le 6d\left(\frac{ne}{3d}\right)^{3d}\left(\frac{d}{n}\right)^{3d} \le ne^{-\Omega(d)}.$$

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Now the random variable D_G is concentrated around its mean. Adding or deleting an edge to G will change D_G by at most $6d$ and using the Azuma-Hoeffding martingale inequality after fixing the number of edges at at most dn we see that

$$\mathbb{P}(D_G \geq \mathbb{E}(D_G) + t) \leq o(1) + \exp\left\{-\frac{2t^2}{72d^3n}\right\}$$

and so putting $t = n^{2/3}$ we see that w.h.p.

$$|A_1| \leq 4d\tau\gamma d^{-\beta/40\alpha}n.$$

Next let $U_{1,c} = U_{1,b} \cup A_1$. Notice now that a vertex $v \in U_0 \setminus U_{1,c}$ only has red edges to $U_{1,c}$ and uncolored edges to A_1. So we now construct $U_1 \supseteq U_{1,c}$ by repeatedly adding vertices $y_1, y_2, \ldots, y_s$ of $U \setminus U_{1,c}$ such that y_j is the lowest numbered vertex not in $Y_j = U_{1,c} \cup \{y_1, y_2, \ldots, y_{j-1}\}$ has at least two neighbors in X_i. This ends with $s \leq 3|A_1|$ by the same argument used to show $r \leq 5|U_{1,a}|$ above. Note that

$$|U_1| \leq \gamma_1 = 7\tau\gamma n = \frac{112\ln^6 d}{\alpha^2 d^{2-3/\alpha}}.$$

This verifies P1–P4 with $i = 0$.

2.2.2 The verification of P1–P4: Constructing U_2

Applying Lemma 2.2 again, with

$$\sigma = \gamma_1 \text{ and } \theta = 3 \text{ and } \delta = 2\theta + \beta/3 \text{ and } \tau = 12/\beta$$

we see that w.h.p.

$$U_2' = \{v \in U_1 : d_{U_1}(v) \geq 6 + \beta/3\} \text{ satsifies } |U_2'| \leq \gamma_2' = 12\gamma_1/\beta = \frac{1344\ln^7 d}{\alpha^3 d^{3-4/\alpha}}. \tag{2.10}$$

We then construct $U_2 \supseteq U_2'$ by repeatedly adding vertices $x_1, x_2, \ldots, x_r$ of $U_1 \setminus U_2'$ such that x_i has at least two neighbors in $U_2' \cup \{x_1, x_2, \ldots, x_{i-1}\}$. This ends with $r \leq 7|U_2'|$ in order that we do not violate the conclusion of Lemma 2.1 with

$$\sigma = 8\gamma_2' = \frac{10752\ln^7 d}{\alpha^2 d^{3-4/\alpha}} \text{ and } \theta = 15/8 \text{ which is applicable since } \left(\frac{10752 \cdot 4 \cdot e\ln^4 d}{15\alpha^3 d^{2-4/\alpha}}\right)^{2-\varepsilon} < \frac{10752\ln^7 d}{2e\alpha^3 d^{3-4/\alpha}}. \tag{2.11}$$

This verifies P1–P4 with $i = 1$.

2.2.3 The verification of P1–P5: Constructing U_i, $i \geq 3$

We now repeat the argument to create the sequence $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$. The value of θ has decreased to $15/8$ (see (2.11)) and $|U_i| \leq (12\gamma/\beta)|U_{i-1}|$, as in (2.10). We choose ℓ so that $|U_\ell| \leq \ln n$. We can easily prove that w.h.p. S contains at most $|S|$ edges whenever $|S| \leq \ln n$, implying P4.

This completes the proof of Part (b) of Theorem 1.2.

3 Theorem 1.3: $G_{n,d}$

We will not change A or B's strategies. We will simply transfer the relevant structural results from $G_{n,d/n}$ to $G_{n,d}$. Some of the unimportant constants will change, but this will not change the verification of the success of the various strategies. We will first do this using Theorem 1.2 under the assumption that $d \leq n^{1/4}$. For larger d we will use Theorem 1.1 and the "sandwiching theorem" of Kim and Vu [15]. This latter analysis is given in Appendix A.

We begin with the configuration model of Bollobás [8]. We have a set W of *points* and this is partitioned into sets $W_1, W_2, \ldots, W_n$ of size d. We define $\phi : W \to [n]$ by $\phi(x) = j$ for all $x \in W_j$. We associate each *pairing* or *configuration* F of W into $|W|/2$ pairs to a multigraph G_F on the vertex set $[n]$. A pair $\{x, y\} \in F$ becomes an edge $(\phi(x), \phi(y))$ of G_F. Now there are $\frac{(dn)!}{(dn/2)!2^{dn/2}}$ pairings and each simple d-regular graph (without loops or multiple edges) arises $(d!)^n$ times as G_F. So for any pair of d-regular graphs G_1, G_2 we have

$$\mathbb{P}(G_F = G_1 \mid G_F \text{ is simple}) = \mathbb{P}(G_F = G_2 \mid G_F \text{ is simple}). \tag{3.1}$$

In order to use this, we need a bound on the probability that G_F is simple.

$$\mathbb{P}(G_F \text{ is simple}) \geq e^{-2d^2}. \tag{3.2}$$

This is the content of Lemma 2 of [9].

It follows from (3.1) and (3.2) that for any graph property $\mathcal{A}$:

$$\varepsilon^{2d^2}\mathbb{P}(G_F \in \mathcal{A}) = o(1) \text{ implies } \mathbb{P}(G_{n,d} \in \mathcal{A}) = o(1). \tag{3.3}$$

We can use the above to estimate $\rho = \mathbb{P}(G_{n,d/n})$ is d regular. We write this as

$$\rho = \mathbb{P}(G = G_{n,d/n} \text{ is } d \text{ regular} \mid \quad |E(G)| = dn/2)\mathbb{P}(|E(G)| = m = dn/2).$$

It is easy to show, using Stirling's approximation, that

$$\mathbb{P}(|E(G)| = m) = \Omega(n^{-1/2})$$

and so we concentrate on the other factor.

Let $N = \binom{n}{2}$. There are $\binom{N}{m} \leq \left(\frac{Ne}{m}\right)^m$, graphs with vertex set $[n]$ and m edges of which

$$\Omega\left(\frac{e^{-2d^2}(dn)!}{(dn/2)!2^{dn/2}(d!)^n}\right) \text{ are } d\text{-regular.}$$

So, since $d = o(n)$,

$$\rho = \Omega\left(\frac{e^{-2d^2}}{n^{1/2}} \cdot \left(\frac{dn}{e}\right)^{dn/2} \cdot \frac{1}{(d!)^n} \cdot \left(\frac{d}{e(n-1)}\right)^{dn/2}\right) = \Omega\left(\frac{d^{dn}}{n^{1/2}e^{dn+2d^2}(d!)^n}\right) = \Omega\left(\left(\frac{1}{10d}\right)^{n/2}\right). \tag{3.4}$$

We need another crude estimate. We make a small modification of Lemma 1 from [9].

Lemma 3.1. *Given* $\{a_i, b_i\}$, $i = 1, 2, \ldots k \leq n/8d$ *then*

$$\mathbb{P}((a_i, b_i) \in E(G_{n,d}), 1 \leq i \leq k) \leq \left(\frac{20d}{n}\right)^k.$$

Proof   Let $\mathcal{G}_d$ denote the set of d-regular graphs with vertex set $[n]$. For $0 \leq t \leq k$ we let

$$\Omega_t = \{G \in \mathcal{G}_d : \{a_i, b_i\} \in E(G), 1 \leq i \leq t \text{ and } \{a_i, b_i\} \notin E(G), t+1 \leq i \leq k\}.$$

We consider the set X of pairs $(G_1, G_2) \in \Omega_t \times \Omega_{t-1}$ such that G_2 is obtained from G_1 by deleting disjoint edges $\{a_t, b_t\}, \{x_1, y_1\}, \{x_2, y_2\}$ and replacing them by $\{a_t, x_1\}, \{y_1, y_2\}, \{b_t, x_2\}$. Given G_1, we can choose $\{x_1, y_1\}, \{x_2, y_2\}$ to be any ordered pair of disjoint edges which are not incident with $\{a_1, b_1\}, \ldots, \{a_k, b_k\}$ or their neighbours and such that $\{y_1, y_2\}$ is not an edge of G_1. Thus each $G_1 \in \Omega_1$ is in at least $(D - (2kd^2 + 1))(D - (2kd^2 + 2))$ pairs, where $D = dn/2$. Each $G_2 \in \Omega_{t-1}$ is in at most $2Dd^2$ pairs. The factor of 2 arises because a suitable edge $\{y_1, y_2\}$ of G_2 has an orientation relative to the switching back to G_1. It follows that

$$\frac{|\Omega_t|}{|\Omega_{t-1}|} \leq \frac{2Dd^2}{(D - (2kd^2 + 1))(D - (2kd^2 + 2d + 2))} \leq \frac{20d}{n}.$$

It follows that

$$\frac{|\Omega_k|}{|\Omega_0| + \cdots + |\Omega_k|} \leq \left(\frac{20d}{n}\right)^k$$

and this implies the lemma. $\square$

3.1 The lower bound

Using (3.2) we can replace (2.3) by

$$e^{2d^2} \exp\left\{(\beta + \gamma + \beta^2 - (2\beta + \gamma)^2/2 + o_d(1)))Dn \ln d\right\} = o(1)$$

for $d \leq n^{1/4}$. After this, we can argue as in the case $G_{n,p}$.

3.2 The upper bound

We first need to prove the equivalent of Lemmas 2.1 and 2.2.

Lemma 3.2. *If* $\theta > 1$ *and*

$$\left(\frac{10\sigma ed}{\theta}\right)^\theta \leq \frac{\sigma}{2e} \tag{3.5}$$

then w.h.p. there does not exist $S \subseteq [n], |S| \leq \sigma n$ *such that* $e(S) \geq \theta|S|$.

Proof

$$\mathbb{P}(\exists S : |S| \leq \sigma n \text{ and } e(S) \geq \theta|S|) \leq \sum_{s=2\theta}^{\sigma n} \binom{n}{s}\binom{\binom{s}{2}}{\theta s}\pi_s$$

where

$$\pi_s = \max_{\substack{X \subseteq \binom{[s]}{2} \\ |X| = \theta s}} \mathbb{P}(E(G_{n,d}) \supseteq X).$$

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It follows from (3.2) that $\pi_s \leq e^{2d^2} \left(\frac{d}{n}\right)^{\theta s}$. If d is small, say $d \leq \ln^{1/3} n$ then we can see from the proof of Lemma 2.1 that

$$\mathbb{P}(\exists S : |S| \leq \sigma n \text{ and } e(S) \geq \theta|S|) \leq O\left(e^{2\ln^{2/3} n} \cdot \frac{d^\theta}{n^{\theta-1}}\right) = o(1).$$

We can therefore assume that $d \geq \ln^{1/3} n$ and then

$$
\begin{aligned}
\sum_{s=3d^2}^{\sigma n} \binom{n}{s}\binom{\binom{s}{2}}{\theta s}\pi_s &\leq e^{2d^2} \sum_{s=3d^2}^{\sigma n} \binom{n}{s}\binom{\binom{s}{2}}{\theta s}\left(\frac{d}{n}\right)^{\theta s} \\
&\leq e^{2d^2} \sum_{s=3d^2}^{\sigma n} \left(e\left(\frac{s}{n}\right)^{\theta-1}\left(\frac{ed}{2\theta}\right)^\theta\right)^s \\
&\leq e^{2d^2} \sum_{s=3d^2}^{\sigma n} 2^{-s} \\
&= o(1).
\end{aligned}
$$

When $s \leq 3d^2$ we use Lemma 3.1. For this we will need to have $\theta s \leq 3\theta d^2 \leq n/8d$. The maximum value of θ is $d^{1/6}\ln^3 d$ and so the lemma can indeed be applied for $d \leq n^{1/4}$. Assuming this, we have

$$
\begin{aligned}
\sum_{2\theta}^{3d^2} \binom{n}{s}\binom{\binom{s}{2}}{\theta s}\pi_s &\leq \sum_{2\theta}^{3d^2} \binom{n}{s}\binom{\binom{s}{2}}{\theta s}\left(\frac{20d}{n}\right)^{\theta s} \\
&\leq \sum_{2\theta}^{3d^2} \left(e\left(\frac{s}{n}\right)^{\theta-1}\left(\frac{20ed}{2\theta}\right)^\theta\right)^s \\
&= o(1).
\end{aligned}
$$

$\square$

Lemma 3.3. *Let σ, θ be as in Lemma 3.2. If*

$$\left(\frac{\sigma ed}{(\delta-2\theta)\tau}\right)^{(\delta-2\theta)\tau} \leq \frac{\sigma}{4e}$$

then w.h.p. there do not exist $S \supseteq T$ such that $|S| \leq \sigma n, |T| \geq \tau s$ and $d_S(v) \geq \delta$ for $v \in T$.

Proof   We first argue that if $d \leq \ln^{1/3} n$ then we prove the lemma by just inflating the failure probability by e^{2d^2} as we did for Lemma 3.2.

We therefore assume that $d \geq \ln^{1/3} n$ and write

$$\mathbb{P}(\exists S \supseteq T, |S| \leq \sigma n, |T| \geq \tau s : \mathbb{P}(|e(T : S \setminus T)| \geq (\delta-2\theta)\tau s) \leq \sum_{s,t} \binom{n}{s}\binom{s}{t}\binom{t(s-t)}{(\delta-2\theta)\tau s}\pi_s$$

where now we have

$$\pi_s = \max_{\substack{X \subseteq T \times (S \setminus T) \\ |X| = (\delta-2\theta)\tau s}} \mathbb{P}(E(G_{n,d}) \supseteq X).$$

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Using (3.2) we write

$$\sum_{s=3d^2/\tau}^{\sigma n} \sum_{t=\tau s}^{s} \binom{n}{s}\binom{s}{t}\left(\frac{t(s-t)}{(\delta-2\theta)\tau s}\right)\pi_s$$

$$\leq e^{2d^2} \sum_{s=3d^2/\tau}^{\sigma n} \sum_{t=\tau s}^{s} \binom{n}{s}\binom{s}{t}\left(\frac{edt}{(\delta-2\theta)\tau n}\right)^{(\delta-2\theta)\tau s}$$

$$\leq e^{2d^2} \sum_{s=3d^2/\tau}^{\sigma n} \sum_{t=\tau s}^{s} \left(2e\left(\frac{s}{n}\right)^{(\delta-2\theta)\tau-1}\cdot\left(\frac{ed}{(\delta-2\theta)\tau}\right)^{(\delta-2\theta)\tau}\right)^s$$

$$\leq e^{2d^2} \sum_{s=3d^2/\tau}^{\sigma n} \sum_{t=\tau s}^{s} 2^{-s}$$

$$= o(1).$$

When $s \leq 3d^2/\tau$ use Lemma 3.1, with the same caveats on the value of d. So,

$$\sum_{s=2\theta}^{3d^2/\tau} \sum_{t=\tau s}^{s} \binom{n}{s}\binom{s}{t}\left(\frac{t(s-t)}{(\delta-2\theta)\tau s}\right)\pi_s$$

$$\leq \sum_{s=2\theta}^{3d^2/\tau} \sum_{t=\tau s}^{s} \binom{n}{s}\binom{s}{t}\left(\frac{t(s-t)}{(\delta-2\theta)\tau s}\right)\left(\frac{20d}{n}\right)^{(\delta-2\theta)\tau s}$$

$$= O\left(\frac{d^{(\delta-2\theta)\tau}}{n^{(\delta-2\theta)\tau-1}}\right) = o(1).$$

$\square$

Remark 1. *We can estimate $\mathbb{P}(\exists \mathcal{C} : |B(\mathcal{C})| \geq \gamma n)$ by multiplying (2.9) by $1/\rho$ and notice that it remains $o(1)$.*

This completes the proof of Theorem 1.3.

4 Theorem 1.4: Random Cubic Graphs

Consider the coloring game on $G_{n,3}$ with three colors. We describe a strategy for B that wins him the game, so $\chi_g(G_{n,3}) \geq 4$ w.h.p. This proves Theorem 1.4: in general $\chi_g(G) \leq \Delta(G) + 1$ where Δ denotes maximum degree, and in particular $\chi_g(G_{n,3}) \leq 4$.

We proceed in two steps. First, we describe a strategy that wins B the game, given the existence of a subgraph H in G satisfying certain conditions. Next, we will prove that w.h.p., a random cubic graph contains such a subgraph.

4.1 The winning strategy

We will say that two vertices are *close* if they are connected by a path of length two or less, and that a path is *short* if some vertex on it is close to both endpoints. (This is not the same as being
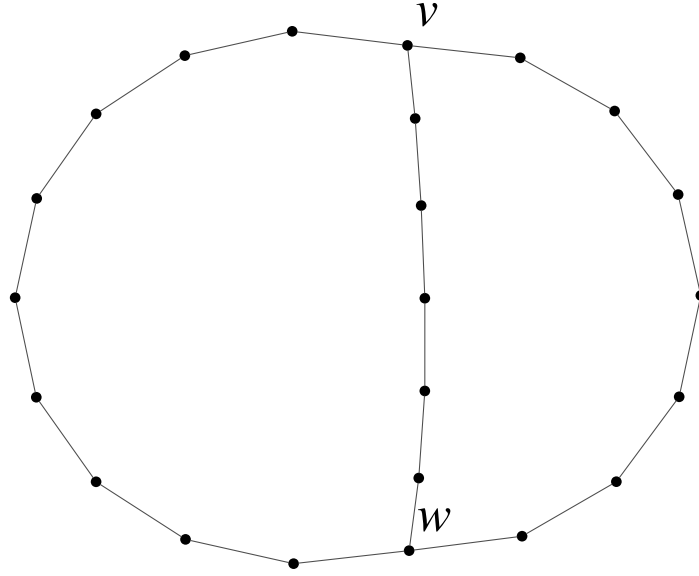
Figure 1: The subgraph H required for Bob's strategy on Random Cubic Graphs.

of length at most four). Vertices that are not close are *far apart* and a path that is not short is *long*. The motivation for this terminology is that coloring a vertex can only have an effect on vertices that are close to it; we will make this precise later on.

We assume first the existence of a subgraph H with the following properties (see Figure 1):

1. H consists of two vertices, v and w, together with three (internally disjoint) paths from one to the other.

2. If A goes first, then the vertex colored by A on her first move is far from H.

3. Each of the paths consists of an even number of edges.

4. No two vertices in H are connected by a short path outside of H (in particular, H is induced).

5. The three paths themselves are all long.

B first plays on the vertex v. Provided A's next move is not on the vertex w, or on the neighbors of v or w, it is close to at most one of the three paths which make up H (this follows from properties 4 and 5). The remaining two paths form a cycle containing v, with no other already colored vertices close to the cycle; by property 3, the cycle is even. Call the vertices around the cycle $(v, v_1, v_2, \ldots, v_{2k-1})$.

Starting from this even cycle, B proceeds as follows. He colors v_2 a different color from v; this creates the threat that on his next move, he will color the third neighbor of v_1 the remaining color, leaving no way to color v_1 and winning. We will call such a move by B a *forcing move at v_1*. A can counter this threat in several ways:

- By coloring v_1 the only remaining viable color.

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- By coloring v_1's third neighbor the same color as either v or v_2.

- By coloring that neighbor's other neighbors in the color different from both v and v_2.

In all cases, A must color some vertex close to v_1, that does not lie on the cycle.

B continues by making a forcing move at v_3: coloring v_4 a different color from v_2. Continuing to play on the even vertices v_{2i}, B makes forcing moves at each odd v_{2i-1}. By property 4 of H, the set of vertices A must play on to counter each threat are disjoint; thus, A's response to each forcing move does not affect the rest of the strategy. By property 2, A's first play does not affect the strategy either.

When B colors v_{2i-2}, this is a forcing move both at v_{2i-3} and at v_{2i-1} (provided Bob chooses a color different both from v_{2i-4} and v). A cannot counter both threats, therefore B wins.

We now account for the remaining few cases. If A colors a neighbor of v or w on her second move, this vertex will be close to all three possible even cycles. However, we know that all three paths in H have even length. Therefore we can still apply this strategy to the even cycle not containing the vertex A colored. Even though it will be close to v or w, we will never need to force at v or at w, because we only force at odd numbered vertices along the path.

Finally, if A colors w itself, then there is no path we can choose that will avoid the vertex. Instead, B picks any of the paths from v to w, and makes forcing moves down that path. Provided that the path is sufficiently long to do so (which follows from property 5), the final move will be a forcing move in two ways, winning the game for B once again.

4.2   Proof of the existence of H

It remains to show that the subgraph H exists w.h.p. (even allowing for A's first move). We will assume G is chosen by adding a random perfect matching to a cycle on n vertices, and find H w.h.p. (we will refer to this Hamiltonian cycle as "the cycle of G" even though other Hamiltonian cycles may exist). That this is a contiguous model to $G_{n,3}$ is well known, see [17]. In the following, let c be a constant; we will later see that we need c to be less than 1 for the proof to hold.

We begin by counting *good* segments of length $m = \lfloor c\sqrt{n} \rfloor$ on the cycle of G, by which we mean those with no internal chords. First of all let X be twice the number of chords that intercept segments of length m or less – these are the only chords that could possibly be internal to a segment of the desired length. X can be written as the sum $X_1 + X_2 + \cdots X_n$, where X_i is the 0-1 indicator for the i-th vertex (call it v_i) to be the endpoint of such a chord. Also, let Y_i denote the length of the smaller of the two segments defined by v_i (this segment stretches from v_i to its partner). Thus

$$\mathbb{P}(Y_i = t) = \begin{cases} \frac{2}{n-1} & 2 \leq t \leq \lfloor (n-1)/2 \rfloor + 1 \\ \frac{1}{n-1} & t = n/2 + 1, \ t \text{ even} \end{cases}$$

Clearly $X_i = 1$ if and only if $Y_i \leq m$, and so

$$\mathbb{E}(X_i) = \frac{2m}{n-1} \text{ and } \mathbb{E}(X) = \frac{2mn}{n-1} \approx 2c\sqrt{n}.$$

In addition, $\mathrm{Var}(X_i) \le \mathbb{E}(X_i)$, and so

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i \ne j} \mathrm{Cov}(X_i, X_j) \le \mathbb{E}(X) + \sum_{i \ne j} \mathrm{Cov}(X_i, X_j).$$

Now

$$\mathrm{Cov}(X_i, X_j) = -\frac{4m^2}{(n-1)^2} + \sum_{t=2}^{m} \mathbb{P}(X_i = 1 \mid Y_j = t)\mathbb{P}(Y_j = t)$$

$$\le -\frac{4m^2}{(n-1)^2} + \frac{2m}{n-3} \cdot \frac{2m}{n-1}$$

$$= \frac{8m^2}{(n-1)^2(n-3)}.$$

Thus,

$$\mathrm{Var}(X) \le \mathbb{E}(X) + \frac{8m^2 n}{(n-1)(n-3)} \approx \mathbb{E}(X).$$

By Chebyshev's inequality,

$$\mathbb{P}(|X - \mathbb{E}(X)| \le \lambda \mathbb{E}(X)) \le \frac{\mathrm{Var}(X)}{\lambda^2 \mathbb{E}(X)^2} \le \frac{2}{\lambda^2 c \sqrt{n}}.$$

Putting $\lambda = n^{-1/5}$ we see that w.h.p. $X \sim 2c\sqrt{n}$.

Consider the n different segments of length m on the cycle of G. Each chord counted by X eliminates at most m of these segments as being good, which leaves $(1 - c^2)n$ segments remaining. We will want non-overlapping good segments; in the worst case that the segments come in intervals of length $m - 1$, we will need to divide by $2m$; even then, there are approximately $(c^{-1} - c)\sqrt{n}$ such good segments w.h.p.

Pick any pair of these segments. If there are exactly 3 chords from one segment to the other, as in Figure 2, then we will construct H as follows (assuming a_i and b_i are the endpoints of the chords, as labeled in Figure 2):

- Set v and w to be a_2 and b_2, respectively.

- The first path from v to w is $(a_2, \ldots, a_1, b_3, \ldots, b_2)$, where the vertices in the ellipses are chosen along the cycle of G.

- The second path from v to w is $(a_2, b_1, \ldots, b_2)$.

- The third path from v to w is $(a_2, \ldots, a_3, b_2)$.

The paths given above require that the three chords are (a_1, b_3), (a_2, b_1), and (a_3, b_2), however, similar paths can be constructed provided that (a_2, b_2) is not one of the chords. In order for H to satisfy properties 3 and 5, we impose conditions on the lengths of the paths $(a_1, \ldots, a_2)$, $(a_2, \ldots, a_3)$, $(b_1, \ldots, b_2)$, and $(b_2, \ldots, b_3)$: they must not be too small, and must have the right parity so that the three paths from v to w have even length. However, these conditions (and the condition that
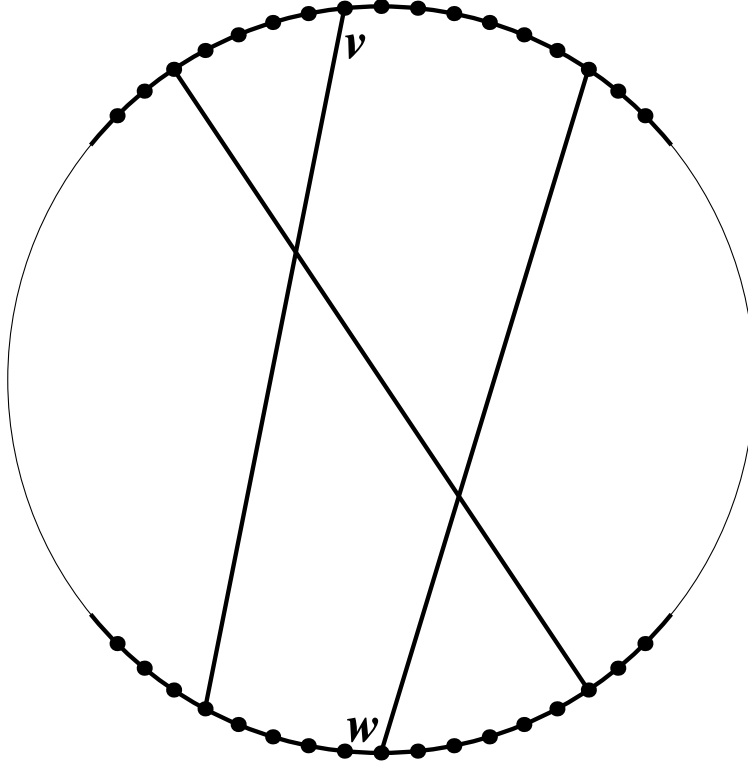
Figure 2: A typical example of the subgraph H found in the Hamiltonian cycle model.

(a_2, b_2) must not be a chord) eliminate only a constant fraction of the possible chords; therefore there are $\Omega(m^6)$ ways to choose the chords.

The probability, then, that a subgraph H can be found between two given good segments, is at least

$$\Omega(m^6) \left(\frac{1}{2n-1} \right)^3 \cdot \left(1 - \frac{m}{n-2m} \right)^{2m}. \tag{4.1}$$

This tends to a constant that does not depend on n.

We now consider the number of pairs of good segments in which we can hope to find this structure. In order to avoid the question of independence, we will divide the $\Omega(\sqrt{n})$ good segments into $\Omega(\sqrt{n})$ pairs, which do not use a good segment more than once. In order to ensure that, should a subgraph H be found, it satisfies property 2, we eliminate all pairs which contain a vertex close to the vertex A chooses on her first move – a constant number of pairs.

It remains to ensure that, should a subgraph H be found, it satisfies property 4: that no two vertices are connected by a short path outside H. To do this, we eliminate all pairs of good segments in which two vertices have chords whose other endpoints are 1 or 2 edges apart. This happens with probability of $O(1/n)$ for any two vertices, and the pair of good segments contains $\binom{2m}{2} \leq 2c^2 n$ pairs of vertices. Therefore with probability at least $(1 - O(1/n))^{2c^2 n}$, which tends to a constant, a pair of good segments satisfies property 4 as a whole, as will any subgraph H contained in it.

We eliminate w.h.p. only a constant fraction of the pairs, and still have $\Omega(\sqrt{n})$ pairs remaining. For each pair, the probability that a subgraph H can be found within it is given by (4.1), and is

constant. The probability that no pair we consider contains a subgraph that we can take to be our H decays exponentially with $\sqrt{n}$. Therefore w.h.p. a subgraph H satisfying properties 1-5 is found.

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A   Adapting Theorem 1.1 to $G_{n,d}$ using the results of [15]

For the random graph $G_{n,p}$, Theorem 1.2 gives results similar to those of Theorem 1.1 (proved in [6]), but which hold on a different range of values of p, and give different constants in the bounds. Just as Theorem 1.3 extends the bounds of Theorem 1.2 to the case of the random d-regular graph $G_{n,d}$, in this appendix we extend Theorem 1.1 to the $G_{n,d}$ case.

Our approach is to use the sandwiching technique developed by Kim and Vu in [15] to adapt the proof of Theorem 1.1. Without changing the strategy used in obtaining the lower bound, we show that each intermediate result used to prove the theorem in [6] continues to hold for random regular graphs $G_{n,d}$ on the range where these can be approximated sufficiently well by random graphs $G_{n,d/n}$.

In order to get the required strength from the Kim-Vu coupling, however, we require $d = np = n^\varepsilon$ for some $\varepsilon \geq \varepsilon_0$, where ε_0 is a small absolute constant. This is not an assumption made in Theorem 1.1. We will assume this throughout this section without further comment.

A.1   Notation

We use Theorem 2 in [15] to get a joint distribution on (H_1, G, H_2): G is d-regular, $H_1 \subseteq G$, $H_1 \subseteq H_2$, and although $G \not\subseteq H_2$, this is almost true in a way we discuss further. The graphs H_1 and H_2 are random graphs with edge probabilities p_1 and p_2, and by judicious choice of parameters we can set $p_1 = p/(1 + \delta)$ and $p_2 = p(1 + \delta)$, where

$$p = \frac{d}{n} \text{ and } \delta = O\left(\left(\frac{\ln n}{d}\right)^{1/3}\right).$$

Constants defined in [6] are in terms of p and we will make this relationship explicit. Of note is the constant

$$\ell_1(p) = \log_b n - \log_b \log_b np - 10 \log_b \ln n$$

where $b = b(p) = \frac{1}{1-p}$.

A.2   Kim-Vu coupling

The construction of the coupling (H_1, G, H_2) in [15] yields $H_1 \subseteq G$ w.h.p., but not $G \subseteq H_2$. As a substitute for such a result, we prove the following lemma.

Lemma A.1. $\Delta(G \setminus H_2) = O(1)$ w.h.p.

Proof   We rely on the bound $\Delta(G \setminus H_2) \leq \Delta(G) - \delta(H_2) + \Delta(H_2 \setminus G)$. Trivially, $\Delta(G) = d$. Part 3 of Theorem 2 in [15] states that w.h.p.

$$\Delta(H_2 \setminus G) \leq \frac{(1 + o(1)) \ln n}{\ln(\delta d/ \ln n)} = \frac{(1 + o(1)) \ln n}{\frac{2}{3} \ln d - \frac{2}{3} \ln \ln n + O(1)} = \frac{3 + o(1)}{2\varepsilon}.$$

We prove that w.h.p. $\delta(H_2) \geq d$. For any vertex v, $\deg_{H_2} v$ follows the binomial distribution $B(n-1, p_2)$. By the Chernoff bound,

$$\mathbb{P}[\deg_{H_2} v < d] \leq \mathbb{P}\left[B(n-1, p_2) < \left(1 - \frac{\delta}{2(1+\delta)}\right)(n-1)p_2\right] \leq e^{-\delta^2/(1+\delta)^2(n-1)p_2/8}.$$

We can simplify the exponent here to

$$-\frac{\delta^2(n-1)p_2}{8(1+\delta)^2} \leq -\frac{\delta^2 d}{10(1+\delta)} = -\frac{\Omega((\ln n)^{2/3} d^{1/3})}{2(1+\delta)} \leq -\Omega(n^{\varepsilon/3}).$$

So $\mathbb{P}[\deg_{H_2} v < d] \leq O(n^{-\Omega(n^{\varepsilon/3})})$ and $\mathbb{P}[\delta(H_2) < d] = o(1)$, completing the proof. $\qquad\square$

A.3 Bounds

We first prove a few auxiliary bounds on the relationship between p, p_1, and p_2, as well as other constants in terms of these probabilities.

Bound A.1.
$$1 \geq \frac{\ell_1(p)}{\ell_1(p_1)} \geq 1 - 2\delta, \text{ and } 1 \leq \frac{\ell_1(p_1)}{\ell_1(p)} \leq 1 + 2\delta.$$

Proof   We first note that if $p = o(1)$ then $\log_{b(p)} np$ decreases with p and so if we let $x = \frac{n}{\log_{np} np \ln^{10} n}$ then,

$$\begin{aligned}
\frac{\ell_1(p_1)}{\ell_1(p)} &= \frac{\log_{b(p_1)} n - \log_{b(p_1)} \log_{b(p_1)} np_1 - 10\log_{b(p_1)} \ln n}{\log_{b(p)} n - \log_{b(p)} \log_{b(p)} np - 10\log_{b(p)} \ln n} \\
&\leq \frac{\log_{b(p_1)} n - \log_{b(p_1)} \log_{b(p)} np - 10\log_{b(p_1)} \ln n}{\log_{b(p)} n - \log_{b(p)} \log_{b(p)} np - 10\log_{b(p)} \ln n} \\
&= \frac{\log_{b(p_1)}(x)}{\log_{b(p)}(x)} = \frac{\ln b(p)}{\ln b(p_1)} \leq \frac{p}{p_1}(1 + \delta p) \leq 1 + 2\delta.
\end{aligned}$$

This proves the second inequality. For the first, we take the reciprocal, and note that $(1 + 2\delta)^{-1} > 1 - 2\delta$. $\qquad\square$

Bound A.2.
$$1 \geq \frac{\ell_1(p_2)}{\ell_1(p)} \geq 1 - 2\delta, \text{ and } 1 \leq \frac{\ell_1(p)}{\ell_1(p_2)} \leq 1 + 2\delta.$$

Proof   Apply Bound A.1 with p_2 in place of p and p in place of p_1, since their relationships are the same. $\qquad\square$

Note now that if $d = n^\theta$ then

$$\frac{\log_b \log_b np}{\log_b n} = \frac{\ln \log_b np}{\ln n} \approx 1 - \theta$$

which implies that

$$\ell_1(p) = (\theta + o(1))\log_b np. \tag{A.1}$$

Bound A.3.
$$(1 - p_1)^{\ell_1(p)} = \frac{\ell_1(p)(\ln n)^{10}}{(\theta + o(1))n} \text{ and } (1 - p_2)^{\ell_1(p)} = \frac{\ell_1(p)(\ln n)^{10}}{(\theta + o(1))n}.$$

Proof. We can write $(1 - p_1)^{\ell_1(p)}$ as

$$\left((1 - p_1)^{\ell_1(p_1)}\right)^{\ell_1(p)/\ell_1(p_1)} = \left(\frac{\log_b np(\ln n)^{10}}{n}\right)^{\ell_1(p)/\ell_1(p_1)} = (1 + o(1))\frac{\log_b np(\ln n)^{10}}{n}.$$

Now use Bound A.1 and (A.1). The proof for p_2 is similar. $\qquad\square$

A.4   Lemmas used for the lower bound in [6]

The strategy used in [6] to prove the lower bound relies on probabilistic assumptions labeled there as Lemmas 2.1 through 2.4. By assuming that those lemmas hold for random graphs (and occasionally referencing the proofs of the original lemmas), we prove that they hold in the random regular case as well. It follows that the lower bound of Theorem 1.1 is valid in the case of $G_{n,d}$ as well, provided our assumption that $d = n^\varepsilon$ holds.

Lemma A.2 (Lemma 2.1 of [6]). *For every $S \subseteq [n]$ with $|S| = \ell_1(p)$, w.h.p.*

$$\ell_1(p)(\ln n)^9 \leq \left|\overline{N}(S)\right| \leq \ell_1(p)(\ln n)^{11}.$$

Proof. For an S as above, $\overline{N}(S) = \overline{N}_G(S) \subseteq \overline{N}_{H_1}(S)$. The distribution of $\overline{N}_{H_1}(S)$ is binomial with mean $n(1 - p_1)^{\ell_1(p)}$, which is at most $O(\ell_1(p)(\ln n)^{10})$ by Bound A.3. We can use Chernoff bounds to get $|\overline{N}_{H_1}(S)| \leq \ell_1(p)(\ln n)^{11}$, which implies the same for $|\overline{N}(S)|$.

The proof of the lower bound is similar, except that we don't have the strict containment $\overline{N}_{H_2}(S) \subseteq \overline{N}_G(S)$. However, by Lemma A.1, any vertex in S has $O(1)$ neighbors in G that it does not have in H_2. Therefore $|\overline{N}_G(S)| \leq |\overline{N}_{H_2}(S)| + O(|S|)$. Because $|S| = \ell_1(p)$, and the Chernoff bound gives $|N_{H_2}| = \Omega(\ell_1(p)(\ln n)^{10})$ w.h.p., this difference will be absorbed in the asymptotic factors. $\qquad\square$

Lemma A.3 (Lemma 2.2 of [6]). *W.h.p. there do not exist $S, A, B \subseteq n$ such that (conditions omitted) and every $x \in B$ has fewer than $ap/2$ neighbors in A (where $a = |A|$).*

Proof. The proof of the corresponding lemma in [6] relies on the distribution to say that the number of neighbors any $x \in B$ has in A is distributed according to the binomial distribution $B(a, p)$, and uses the Chernoff bound $\mathbb{P}[B(a, p) \leq ap/2]^{b_1} \leq e^{-ab_1p/8}$.

The number of edges between x and A is bounded below by the number of such edges in the graph H_1, which is distributed according to $B(a, p_1)$. So we replace the bound above by

$$\mathbb{P}[B(a, p_1) \leq ap/2] = \mathbb{P}\left[B(a, p_1) \leq \frac{ap_1(1 + \delta)}{2}\right] \leq \mathbb{P}\left[B(a, p_1) \leq ap_1\left(1 - \frac{1 - \delta}{2}\right)\right].$$

By the Chernoff bound, this is at most $e^{-ap_1(1-\delta)^2/8} \leq e^{-(1-o(1))ab_1p/8}$, and the argument of [6] still goes through. $\qquad\square$

Lemma A.4 (Lemma 2.3 of [6]). *Let $a = 2000\varepsilon^{-2}$. W.h.p. there do not exist sets of vertices $S, T_1, \ldots, T_a$ such that (conditions omitted) and $N(S) \cap T_i = \emptyset$ for $i = 1, \ldots, a$.*

Proof. If such sets exist in the graph G, then they will still exist when we lose some edges in passing to the graph H_1. Examining the proof in [6] we see that all it requires is to consider the following factor which inflates the probability estimate they use:

$$\left(\frac{1-p_1}{1-p}\right)^{(2000/\varepsilon^2)(\varepsilon^2\ell_1(p)^2)/(21^2\cdot 2)} \leq \left(\frac{1}{1-p}\right)^{\ell_1(p)^2/3}.$$

Since $\ell_1(p) < \log_b n$, where $b = \frac{1}{1-p}$, this is at most a factor of $n^{\ell_1(p)/3}$. In the original proof, the conclusion followed from

$$\left(\frac{\ell_1(p)}{n}(\ln n)^{2000/\varepsilon}\right)^{\ell_1(p)} = o(1).$$

With the extra factor we get from working in H, this becomes

$$\left(\frac{\ell_1(p)}{n^{2/3}}(\ln n)^{2000/\varepsilon}\right)^{\ell_1(p)}$$

which is $o(1)$ just as readily. $\qquad\square$

Lemma A.5 (Lemma 2.4 of [6]). *Let $t = \frac{n}{\ell_1(p)(\ln n)^7}$. W.h.p. there do not exist pairwise disjoint sets of vertices $S_1, \ldots, S_t, U$, such that (conditions omitted) and $|U \cap \overline{N}(S_i)| \leq \ell_1(p)(\ln n)^8$ for $i = 1, \ldots, t$.*

Proof. Suppose such sets exist in the graph G. By Lemma A.1 each vertex of S_i has at most $O(1)$ neighbors in G that are not in H_2; therefore in passing to the graph H_2, $|U \cap \overline{N}(S_i)|$ will be at most $\ell_1(p)(\ln n)^8 + O(\ell_0(p))$ where each $|S_i| = \ell_0(p) = \ell_1(p) + C/p$ for some constant C (a fact we will use again). Since $\ell_1(p) = \frac{\ln n}{(\theta+o(1))p}$, the new size of $|U \cap \overline{N}(S_i)|$ is still $(1+o(1))\ell_1(p)(\ln n)^8$. There is room in the argument of [6] to prove this lemma for intersections of this size as well. Therefore we will proceed by arguing that w.h.p. sets such as $S_1, \ldots, S_t, U$ do not exist in H_2.

The proof in [6] hinges upon the claim that $(1-p)^{\ell_0(p)} = \Omega((1-p)^{\ell_1(p)})$. So it suffices to prove that $(1-p_2)^{\ell_0(p)} = \Omega((1-p)^{\ell_1(p)})$. We split $(1-p_2)^{\ell_0(p)}$ into factors $(1-p_2)^{\ell_1(p)}$ and $(1-p_2)^{C/p}$. By Bound A.3, the first factor is $\Omega((1-p)^{\ell_1(p)})$. The second factor is no less than $(1-p_2)^{C/p_2}$, which stays in $(e^{-C}, 4^{-C}]$ as p_2 ranges over $(0, 1/2]$, so it is effectively a constant. $\qquad\square$

A.5 Lemmas used for the upper bound in [6]

As in the lower bound, the strategy used in [6] to prove the upper bound relies on some properties that hold w.h.p. in $G_{n,p}$. Again, we apply the Kim-Vu Sandwich Theorem [15, Theorem 2] to show that the same properties hold in $G_{n,d}$ as well.

The first player's strategy described in [6] is simple — she chooses an uncolored vertex with minimal number of available colors and then colors it with an arbitrary (available) color. We present some notation used in [6] during the analysis of the strategy. Given a (partial) coloring $\mathcal{C}$ and a vertex v let $\alpha(v, \mathcal{C})$ be the number of available colors for v in $\mathcal{C}$. For a constant $\alpha > 3$ define

$$\beta_G = \alpha \frac{n(np)^{-1/\alpha}}{\log_b np}, \qquad\qquad \gamma_G = \frac{10n \ln n}{\beta_G}.$$

and

$$\mathcal{B}(\mathcal{C}) = \{v \mid \alpha(v) \le \beta_G/2\}.$$

The first lemma states that there are not many vertices with few available colors. We show that the same is also true in $G_{n,d}$.

Lemma A.6 (Lemma 3.1 of [6])**.** *W.h.p. for all collections $\mathcal{C}$,*

$$|\mathcal{B}(\mathcal{C})| \le \gamma.$$

Proof   Here we can just use Remark 1. $\qquad\square$

Lemma A.7 (Lemma 3.2 of [6])**.** *W.h.p. every subset S of $G_{n,p}$ of size s spans at most $\phi = \phi(s) = (5ps + \ln n)s$ edges.*

Proof. The proof in [6] actually gives the result for $\phi_2 = (4.5ps + 0.9\ln n)s$. Thus, applying Lemma 3.2 of [6] to H_2 gives that w.h.p. every set S of size s spans at most $(4.5p_2s + 0.9\ln n)s$ edges. Since w.h.p. every vertex of G touches at most $O(1)$ edges not in H_2, we have that w.h.p. the number of edges spanned by S in G is bounded by

$$(4.5p_2s + 0.9\ln n + O(1))s = (4.5ps(1 + o(1)) + 0.9\ln n + O(1))s \le (5ps + \ln n)s$$

as required. $\qquad\square$